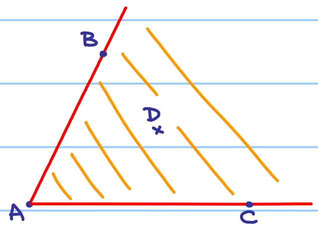
3.4 Axioms of Congruence for Angles

Recall :

An angle is the union of two rays r_{AB} and r_{AC} originating at the same point, its vertex, and not lying on the same line We denote the angle by $\angle BAC$ or $\angle CAB$.

Furthermore, the interior of an angle $\angle BAC$ consists of all points D such that D and C are on the same side of the line AB, and D and B are on the same side of the line AC.



We also postulate an undefined notion of congruence, which is a relation between two angles α and β, denoted by $\alpha \cong \beta$.

Suppose that $\mathscr{A}$ is the set of all angles.

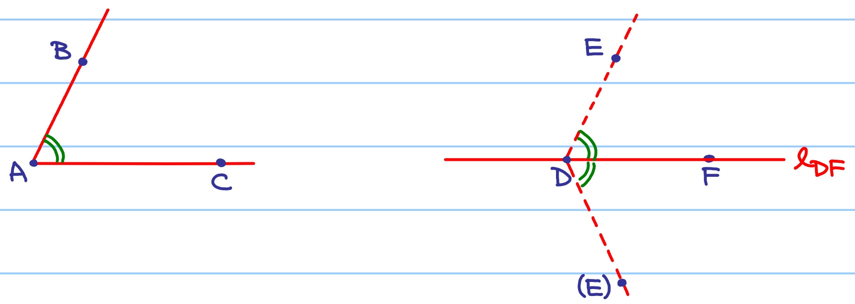
Definition 3.4.1

Let $\mathcal{C} \subseteq \mathscr{A} \times \mathscr{A}$ If $(\alpha, \beta) \in \mathcal{C}$, then α is said to be congruent to β, and we denote it by $\alpha \cong \beta$.

Then, we impose axioms on $\mathcal{C}$ such that it behaves as we expect.

Axioms of Congruence for Angles :

(C.4) Given an angle $\angle BAC$ and given a ray r_{DF}, there exists a unique ray r_{DE} on a given side of the line ℓ_{DF} such that $\angle BAC \cong \angle EDF$.

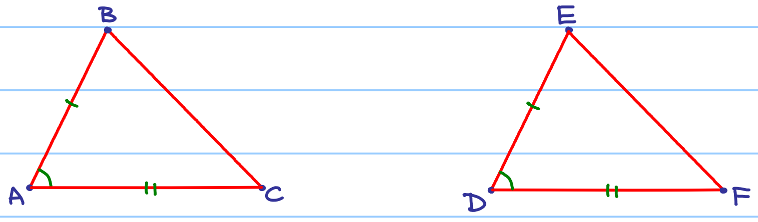


(C.5) For any three angles α, β, γ, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$

Every angle is congruent to itself.

(C.6) (SAS) Given triangles ABC and DEF, suppose that $AB \cong DE$, $AC \cong DF$ and $\angle BAC \cong \angle EDF$.

Then $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.



Remark: (C.4) acts as "transporter of angles".

(C.5) says that congruence is an equivalence relation on $\mathcal{A}$.

Compare to (C.1) and (C.2)

However, (C.6) is unlike (C.3). It turns out

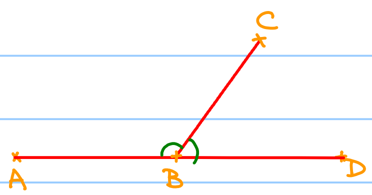
- (C.6) implies a result similar to (C.3)
- (C.6) is postulated, which shows the insufficiency of Euclid's proof of proposition 1.4 using method of superposition.

Recall: No definition of "straight angle".

Problems:

- $\mathcal{A}$: set of angles, is it possible to define $+ : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$?

  $\angle ABC$ "+" $\angle CBD$ may not make sense



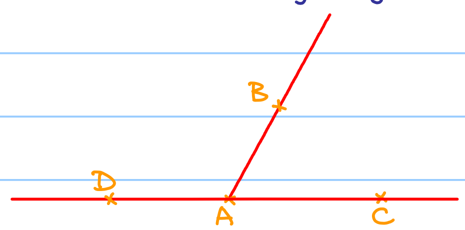
- How to reformulate C.N.2 and C.N.3 (for angles)?

(If $\beta \cong \beta'$ and $\gamma \cong \gamma'$, then $\beta + \gamma \cong \beta' + \gamma'$ and $\beta - \gamma \cong \beta' - \gamma'$)

Definition 3.4.2

If $\angle BAC$ is an angle, and if D is a point on the line AC on the other side of A from C, then $\angle BAC$ and $\angle BAD$ are said to be supplementary.

(Remark: Give a definition to "sum of two angles is two right angles".)



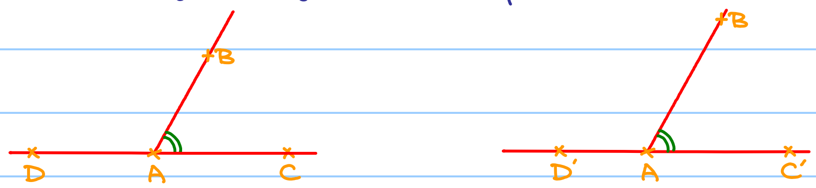
Proposition 3.4.1

Let $\angle BAC$ and $\angle BAD$ be supplementary angles and $\angle BAC \cong \angle B'A'C'$

Then, $\angle B'A'C'$ and $\angle B'A'D'$ are supplementary angles if and only if $\angle BAD \cong \angle B'A'D'$.

Remark: "$\Rightarrow$" is regarded as a replacement of C.N.3 for the case "$\beta = 180°$"

"$\Leftarrow$" is regarded as a replacement of C.N.2 for the case "$\beta + \gamma = 180°$"



proof:

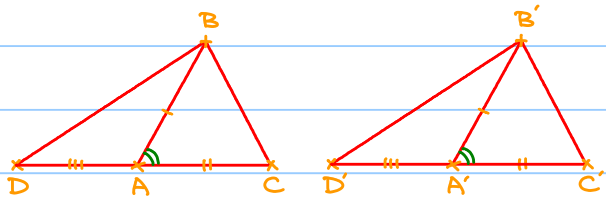
"$\Rightarrow$": By choosing another B', C' and D' if necessary, we assume $AB \cong A'B'$, $AC \cong A'C'$ and $AD \cong A'D'$ (By C.1).

$\triangle ABC \cong \triangle A'B'C' \Rightarrow BC \cong B'C'$ and $\angle BCA \cong \angle B'C'A'$ (C.6, SAS)

$CA \cong C'A'$ and $AD \cong A'D' \Rightarrow CD \cong C'D'$

$\triangle BCD \cong \triangle B'C'D' \Rightarrow BD \cong B'D'$ and $\angle BDA \cong \angle B'D'A'$ (C.6, SAS)

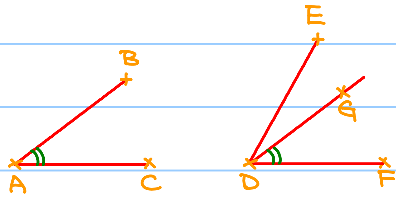
$\triangle BDA \cong \triangle B'D'A' \Rightarrow \angle BAD \cong \angle B'A'D'$ (C.6, SAS)



"$\Leftarrow$": Exercise!

Definition 3.4.3

Given two angles $\angle BAC$ and $\angle EDF$. We say $\angle BAC$ is less than $\angle EDF$, denoted by $\angle BAC < \angle EDF$, if there exists a ray r_{DG} in the interior of $\angle EDF$ such than $\angle BAC \cong \angle GDF$.



Exercise 3.4.1

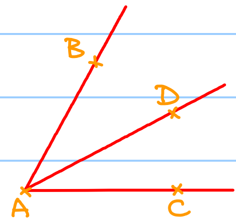
Prove that there exists a ray r_{DG} in the interior of $\angle EDF$ such than $\angle BAC \cong \angle GDF$ if and only if there exists a ray $r_{DG'}$ in the interior of $\angle EDF$ such than $\angle BAC \cong \angle G'DE$.

Definition 3.4.4

If $\angle BAC$ is an angle, and if a ray r_{AD} lies in the interior of $\angle BAC$, then $\angle BAC$ is said to be the sum of $\angle DAC$ and $\angle BAD$

Remark: Only define $\angle DAC + \angle BAD$ when it makes sense.

Furthermore $\angle DAC, \angle BAD < \angle BAC$

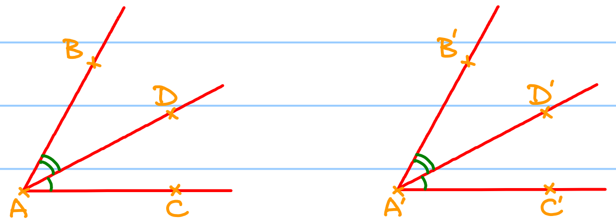


Proposition 3.4.2

Suppose $\angle BAC$ is an angle, and the ray r_{AD} is in the interior of $\angle BAC$.

Suppose $\angle D'A'C' \cong \angle DAC$ and $\angle B'A'D' \cong \angle BAD$, and the rays $r_{A'B'}$ and $r_{A'C'}$ are on opposite sides of the line $A'D'$. Then the ray $r_{A'B'}$ and $r_{A'C'}$ form an angle, and $\angle B'A'C' \cong \angle BAC$, and the ray $r_{A'D'}$ is in the interior of $\angle B'A'C'$

Remark: It is regarded as a replacement of C.N.2 for the case "$\beta + \gamma < 180°$"



proof:

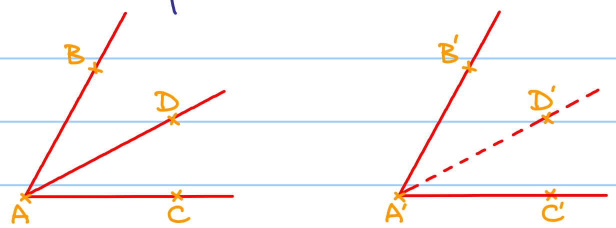
see prop. 8.4(a) in [2]

Exercise 3.4.2

Given $\angle BAC \cong \angle B'A'C'$ and a ray r_{AD} in the interior of $\angle BAC$.

Prove that there exists a ray $r_{A'D'}$ in the interior of $\angle B'A'C'$ such that $\angle DAC \cong \angle D'A'C'$ and $\angle BAD \cong \angle B'A'D'$.

Remark: It is regarded as a replacement of C.N.3 for the case "$\beta + \gamma < 180°$".



Exercise 3.4.3

Prove that

a) If $\alpha \cong \alpha'$ and $\beta \cong \beta'$, then $\alpha < \beta \Leftrightarrow \alpha' < \beta'$.

b) (i) $\alpha < \beta$ and $\beta < \gamma \Rightarrow \alpha < \gamma$

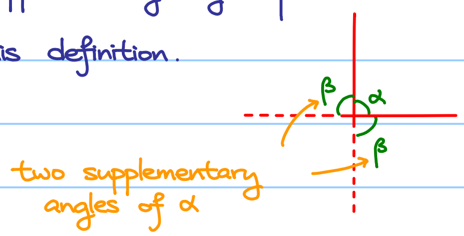
(ii) Given two angles α and β, exactly one of the following holds: $\alpha < \beta$; $\alpha \cong \beta$; $\alpha > \beta$.

Remark: Hence $<$ defines a strict total order relation on $\mathcal{A}/\cong$.

Definition 3.4.5

A right angle is an angle α that is congruent to one of its supplementary angles β.

Remark: The following proposition guarantees no ambiguity of this definition.



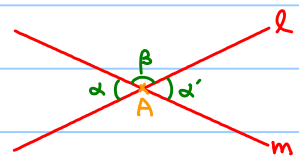
Definition 3.4.6

Let ℓ and m be two non parallel lines and let A be the intersection point of ℓ and m

Vertical angles are defined by opposite rays originated from A on ℓ and m.

Corollary 3.4.1

Vertical angles are congruent (i.e. $\alpha \cong \alpha'$).



proof:

Direct application of proposition 3.4.1.

Proposition 3.4.3

Any two right angles are congruent to each other.

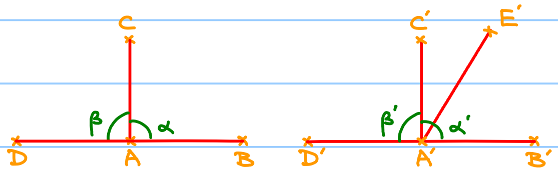
proof:

By assumption, $\alpha \cong \beta$ and $\alpha' \cong \beta'$, we claim $\alpha \cong \alpha'$.

Suppose the contrary.

WLOG, let $\alpha < \alpha'$,

then there exists a ray $r_{A'E'}$ in the interior of α' such that $\angle E'A'B' \cong \alpha$.



Exercise: Show that the ray $r_{A'C'}$ is in the interior of $\angle E'A'D'$

$\Rightarrow \beta' < \angle E'A'D'$

$\angle E'A'B' \cong \alpha \Rightarrow \angle E'A'D' \cong \angle CAD = \beta$ (Prop. 3.4.1)

$\therefore \alpha' \cong \beta' < \angle E'A'D' \cong \angle CAD = \beta \cong \alpha$ which contradicts to the assumption $\alpha < \alpha'$.

3.5 Hilbert Planes

Definition 3.5.1

A Hilbert plane is a given set of points together with certain subsets called lines, and undefined notions of betweenness, congruence of line segments, and congruence of angles that satisfy the axioms (I1)-(I3), (B1)-(B4) and (C1)-(C6).

(Do NOT include (P)!)

Main goal: How much of Euclid's Book I we can recover in a Hilbert plane?

Theorem 3.5.1

Euclid's proposition 1-28, except 1 and 22, in Book 1 can be proved in arbitrary Hilbert plane.

proof:

see section 10 in [2]

Idea: $\mathbb{R}^2$, Klein Disk, Poincaré Disk ... are all Hilbert planes, so all of them inherit all properties of Hilbert planes.

3.6 Intersections of Lines and Circles

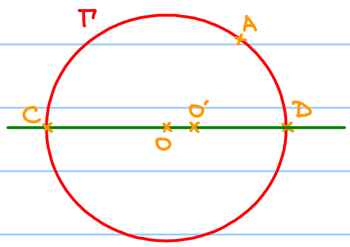
Definition 3.6.1

Given distinct points O, A, the circle Γ with center O and radius OA is the set $\Gamma = \{B : OA \cong OB\}$.

Proposition 3.6.1

Let Γ be a circle with center O and radius OA. Then, the center is uniquely determined.

proof:



Let O and O' be centers of a circle Γ and $O \neq O'$.

Let OA be a radius of Γ.

Let ℓ be a line passing through O and O'.

Then ℓ meet Γ at two points C and D that satisfies

$C*O*D$ and $OC \cong OD$ (Why? Think axiom C1)

Note that O' is also a center, so $O'C \cong O'D$ and $C*O'*D$

By axiom C3, we have three cases:

1) $O'*C*O$: Impossible, since $C*O*D$ and $C*O'*D$

2) $C*O*O'$: It implies $O*O'*D$. Therefore $OC < O'C \cong O'D < OD$ which contradicts to $OC \cong OD$.

3) $C*O'*O$: It implies $O'*O*D$ Therefore $O'C < OC \cong OD < O'D$ which contradicts to $O'C \cong O'D$.

$\therefore O = O'$

Definition 3.6.2

Let Γ be a circle with center O and radius OA.

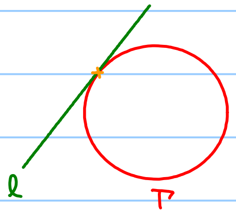
A point B is said to be an interior point of Γ if $B = O$ or $OB < OA$.

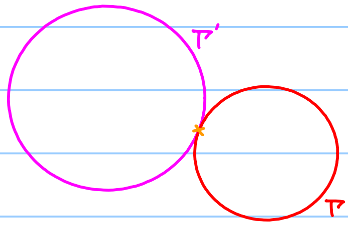
A point C is said to be an exterior point of Γ if $OC > OA$

Definition 3.6.3

A line ℓ is said to be tangent to a circle Γ if ℓ and Γ meet at exactly one point.

A circle Γ is said to be tangent to another circle Γ' if Γ and Γ' meet at exactly one point.





Proposition 3.6.2

Let Γ be a circle with center O and radius OA.

Let ℓ be a line passing through A. Then,

$\ell \perp OA$ if and only if ℓ is tangent to Γ at A.

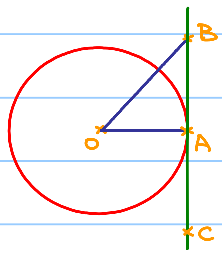
proof:

"$\Rightarrow$": Let $B \in \ell$ and $B \neq A$. Claim $OB > OA$ which shows that B is an exterior point of Γ.

$\angle OAC$ is a right angle

By prop. (I.16), $\angle OBA$, $\angle AOB$ are less than a right angle

By prop. (I.19), $OB > OA$.



"$\Leftarrow$": Note that $\ell \neq \ell_{OA}$, otherwise $|\ell \cap \Gamma| = 2$

Therefore, $O \notin \ell$ and by prop. (I.11), we can find a point $B \in \ell$ such that $OB \perp \ell$.

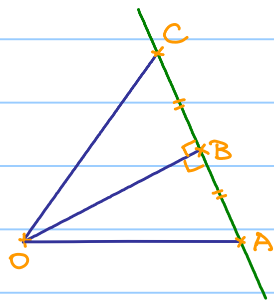
Claim : $B = A$.

Suppose the contrary, $B \neq A$.

Take C on the other side of B from A such that $AB \cong CB$.

By axiom (C.4), $\triangle OBA \cong \triangle OBC$ and so $OA \cong OC$

$\Rightarrow$ $(C \neq A)$ $C \in \Gamma$ (Contradiction)



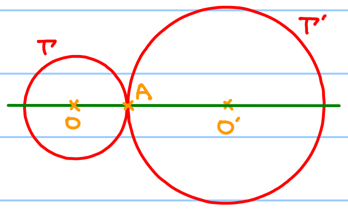
Exercise 3.6.1

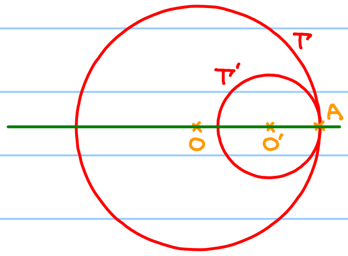
If a line ℓ contains a point A of a circle Γ, but not tangent to Γ, prove that ℓ meets Γ at exactly two points.

Proposition 3.6.3

Let O, O' and A be three distinct points.

Let Γ and Γ' be circles with centers O and O' with radii OA and $O'A$ respectively. Then, O, O' and A are collinear if and only if Γ is tangent to Γ' at A.





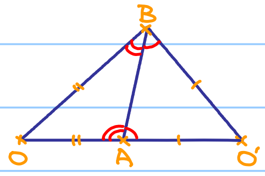
proof:

"⇒": Suppose the contrary. Let $B \neq A$ and $B \in \Gamma \cap \Gamma'$

Firstly, $B \notin l_{OO'}$. Otherwise B must be on the opposite side of A from O.

Then $O'A \cong O'B$ implies $O = O'$ (Contradiction).

Case 1: $O * A * O'$



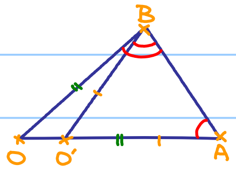
By prop. (I.5), $\angle OAB \cong \angle OBA$ and $\angle O'AB \cong \angle O'BA$

Note that $\angle OAB$ and $\angle O'AB$ are supplementary.

By proposition 3.4.1, $\angle OBA$ and $\angle O'BA$ are supplementary.

Therefore O, B and O' are collinear.

Case 2: $O * O' * A$ (as well as $O' * O * A$)



By prop. (I.5), $\angle OBA \cong \angle OAB \cong \angle O'BA$

By axiom (C.4), $\triangle OBA \cong \triangle O'BA$ and so $OA \cong O'A$

which contradicts to $O * O' * A$.

"$\Leftarrow$". Suppose the contrary, O, O' and A are not collinear.

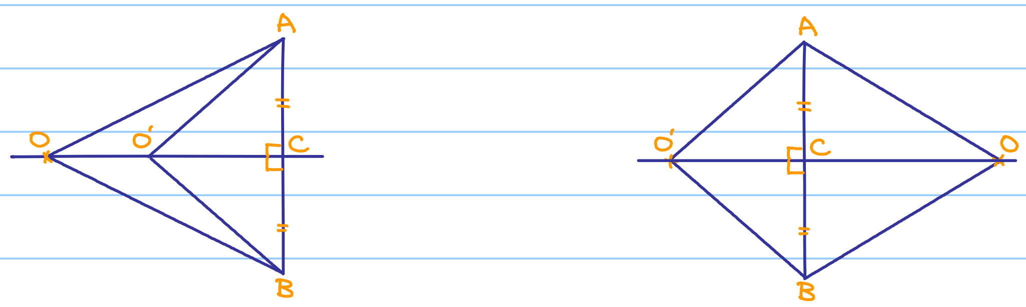
Therefore, $A \notin l_{OO'}$ and by prop. (I.11), we can find a point $C \in l_{OO'}$ such that $OC \perp l_{OO'}$.

Let B be a point on the opposite side of A from C such that $CA \cong CB$.

By axiom (C.4), $\Delta OCA \cong \Delta OCB$ and so $OA \cong OB$

By axiom (C.4), $\Delta O'CA \cong \Delta O'CB$ and so $O'A \cong O'B$

Therefore $A, B \in \Gamma \cap \Gamma'$ but $A \neq B$ which contradicts to that Γ is tangent to Γ' at A.



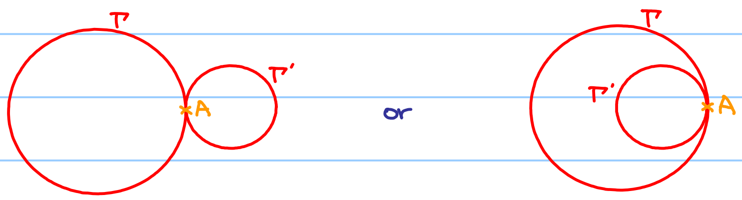
Exercise 3.6.2

Show that if two circles meets at A but they are not tangent, then they have exactly two intersection points.

(Hint: As the construction above, let $A, B \in \Gamma \cap \Gamma'$ with $A \neq B$. Using prop. (I.7) to prove there is no more intersection point.)

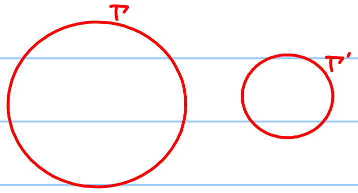
Exercise 3.6.3

If two circles Γ and Γ' are tangent to each other at a point A, show that $\Gamma' \setminus \{A\}$ lies either entirely inside Γ or entirely outside Γ.



Let Γ and Γ' be circles.

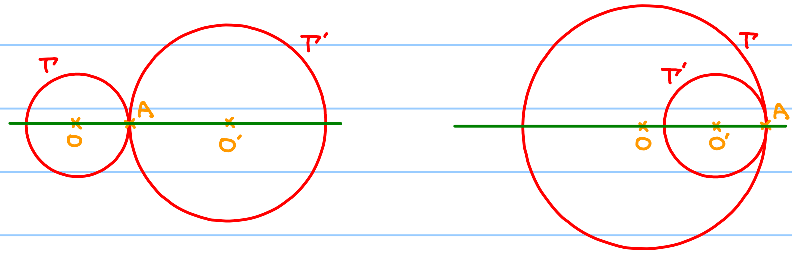
$\Gamma \cap \Gamma' = \phi$



$\Gamma \cap \Gamma' \neq \phi$: Case 1 : Γ and Γ' are tangent to each other at a point A, i.e. $\Gamma \cap \Gamma' = \{A\}$.

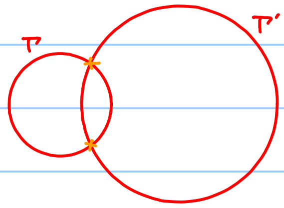
Exercise 3.6.3 $\Rightarrow$ Only two possibilities as shown below.

Proposition 3.6.3 $\Rightarrow$ O, O', A are collinear.



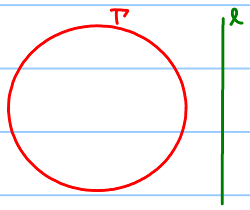
Case 2 : Γ and Γ' are not tangent to each other, i.e. $|\Gamma \cap \Gamma'| > 1$

Exercise 3.6.2 $\Rightarrow |\Gamma \cap \Gamma'| = 2$



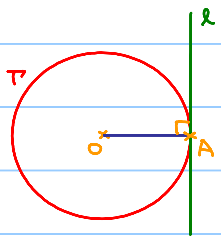
Let Γ be a circle and ℓ be a line.

$\Gamma \cap \Gamma' = \phi$



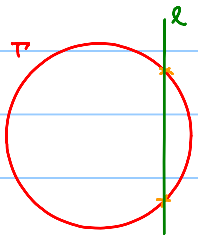
$\Gamma \cap \ell \neq \phi$: Case 1 : Γ and ℓ are tangent to each other at a point A, i.e. $\Gamma \cap \ell = \{A\}$

Proposition 3.6.2 $\Rightarrow$ $\ell \perp OA$



Case 2 : Γ and ℓ are not tangent to each other, i.e. $|\Gamma \cap \ell| > 1$

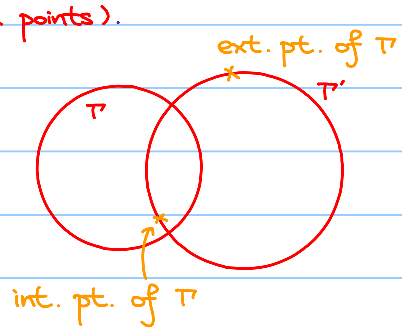
Exercise 3.6.1 $\Rightarrow |\Gamma \cap \ell| = 2$



From the above, we know that a line and a circle, or two circles, can be tangent, or if they meet but are not tangent, they will meet at exactly two points. However, there is nothing here to guarantee they will actually meet if they are in suitable position.
Here, we impose an additional axiom:

Circle-Circle Intersection Property:

(P) Given two circles Γ and Γ', if Γ' contains at least one interior point and at least one exterior point of Γ, then Γ and Γ' will meet (at exactly 2 points).



Proposition 3.6.4 (Line-circle intersection property)

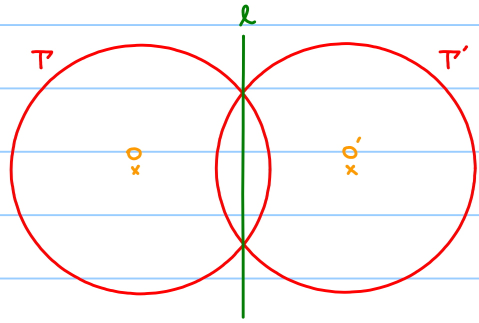
In a Hilbert plane with axiom (E), if a line ℓ contains an interior point A of a circle Γ, then ℓ and Γ will meet (at exactly 2 points).

proof:

See prop. 11.6 in [2]

Idea: Construct a circle Γ' so that Γ and Γ' will meet at 2 points (by axiom (E)).

Then, show that $\Gamma \cap \ell = \Gamma \cap \Gamma'$



Proposition 3.6.4 (Proposition I.1)

It is possible to construct an equilateral triangle on a given line segment.

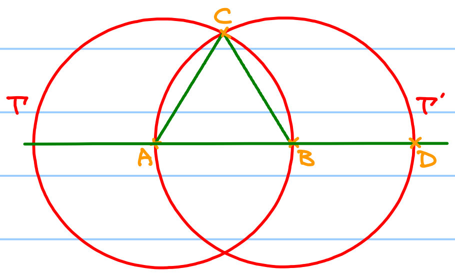
proof.

Only missing part in the original proof.

Note: Γ' contains A (int. pt. of Γ)

and contains D (ext. pt. of Γ)

$\therefore \Gamma \cap \Gamma' \neq \phi$ by axiom (E)



Theorem 3.6.1

Euclid's proposition 1 and 22, in Book 1 are valid in a Hilbert plane with the extra axiom (E).

3.7 Euclidean Plane

Definition 3.7.1

A Euclidean Plane is a Hilbert plane satisfying the additional axioms (E), the circle-circle intersection property, and (P), Playfair's axiom.

(so, it satisfies (I1)-(I3), (B1)-(B4), (C1)-(C6), (E) and (P).)

Theorem 3.7.1

Euclid's proposition 29-34, in Book I are valid in a Hilbert plane with the extra axiom (P)

In chapter 5 of [2], theory of area can be developed on a Hilbert plane with the extra axiom (P).

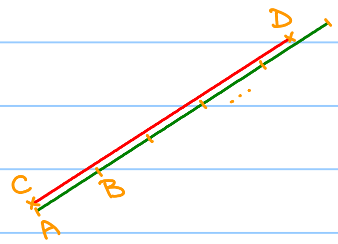
Theorem 3.7.2

Euclid's proposition 35-48, in Book I are valid in a Hilbert plane with the extra axiom (P).

Hence, all propositions in Euclid's Book I are valid in an Euclidean Plane.

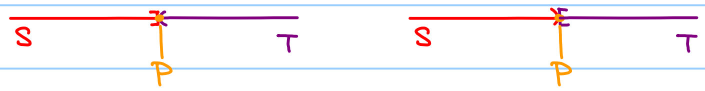
Archimede's Axiom:

(A) Given two line segments AB and CD, there exists a natural number n such that n copies of AB will be greater than CD.



Dedekind's Axiom:

(D) Suppose the points of a line are divided into two nonempty subsets S and T such that no point of S is between two points of T, and vice versa. Then there exists a unique point P such that for any $A \in S$ and any $B \in T$, either $A = P$ or $B = P$ or the point P is between A and B.



Remark: (D) is very strong in a sense that (D) implies (A) and (E).

Theorem 3.7.3

A Hilbert plane satisfying (P) and (D) is isomorphic to $\mathbb{R}^2$.